A COMPARISON OF PLAY-THE-WINNER AND VECTOR-AT-A-TIME SAMPLING FOR SELECTING THE BETTER OF TWO BINOMIAL POPULATIONS WITH RESTRICTED PARAMETER VALUES*

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In a recent paper we have compared two sampling rules for choosing the "better" of two binomial populations, [1]. The model considered was one in which either of two treatments, A or B, was tested with success or failure (the only two possible results under our formulation) being evident without any delay. The probabilities of success for A and B are denoted by p and p', respectively, with $p > p'$. The object in conducting the trials was to determine the better treatment with probability greater than a preassigned P^* of being correct when the difference $p - p'$ is greater than a given value $\Delta^* > 0$. This corresponds to a commonly accepted formulation for ranking and selection problems, [2]. Two sampling rules were analyzed; the first being the "play the winner rule," (PWR), in which a success on a particular treatment generates a further trial on the same treatment, while a failure on a particular treatment generates a trial on the other treatment. The second rule is "vector at a time," (VT), in which treatments A and B are assigned in alternating fashion; i.e.,ABABAB.... The termination rule in [1] is based on the cumulative difference of successes between the two treatments. When the difference reaches a critical value r_c, trials are stopped and the treatment with the greater number of successes is declared the better one. The parameter r_c is expressible in terms of the given P^* and Δ^*.

Our criterion for deciding which sampling rule is better is based on the expected number of failures on the poorer treatment that could have been prevented by knowing and using the better treatment throughout the trial period. This criterion is suggested (for example) by ethical questions in the conduct of clinical trials, in which one wishes to minimize the use of the poorer drug or poorer treatment, while arriving at a statistically valid decision.

In reference [1] we allowed the parameters p and p' to be arbitrary in the interval $(0, 1)$. In the present paper we consider the consequences of having partial knowledge of p and p'. Specifically we will assume that the experimenter has some prior knowledge of a given interval $M = (M_e, M_u)$ such that p and p' are known to satisfy $M_e \leq p', p \leq M_u$. No further assumptions will be made, and in particular we will not carry out a Bayesian analysis, although this might be of independent interest. In order to calculate the critical parameter r_c, it is necessary to define a least favorable configuration, that is, to assume values of p and p' that are the least favorable values consistent with the constraint that $PCS \geq P^*$ whenever $p - p' \geq \Delta^*$. An interesting result found in [1] is that for $M_e = 0$, $M_u = 1$ the least favorable configurations for PW- and VT-sampling are different. For VT-sampling the least favorable configuration is

$$p'_{LF}(VT) = (1 - \Delta^*)/2 \; ; \; p_{LF}(VT) = (1 + \Delta^*)/2 \quad (1)$$

while for PW-sampling the least favorable configuration is very closely given by

$$p'_{LF}(PW) = 1 - \Delta^* \; ; \; p_{LF}(PW) = 1 \, . \quad (2)$$

When $M \neq (0, 1)$ the least favorable configurations for both sampling schemes can change from those given above. It is this feature of the problem that suggests that we can obtain different results from those of [1].

Let $\bar{p}_0 = 1 - (\Delta^*/2)$ and assume that $\Delta^* \leq 1/2$. We shall consider four possibilities for $M = (M_e, M_u)$. These are:

1. $0 < M_e < M_u < (1 + \Delta^*)/2$

2. $M_e < (1 + \Delta^*)/2 \leq M_u < \bar{p}_0$

3. $(1 + \Delta^*)/2 < M_e \leq M_u < \bar{p}_0$

4. $(1 + \Delta^*)/2 < M_e \leq \bar{p}_0 \leq M_u$.

We omit consideration of the possibility $\bar{p}_0 < M_e < M_u \leq 1$ since this is of little practical interest. In order to apply the methodology of [1] we must first determine the least favorable configurations corresponding to the four cases enumerated above. There are two fundamental relations which allow one to calculate critical parameters r_c and s_c for PW- and VT-sampling, respectively, that were proved in [1]. If p' and p are assumed known, then in order to insure a probability of correct selection equal to P^* with PW-sampling we first compute the root in r of

$$\lambda^r = \frac{1}{2qP^*} \left[\left(\frac{q + q'}{2}\right) - \sqrt{\left(\frac{q + q'}{2}\right)^2 - 4qq'P^*(1 - P^*)} \right], \qquad (3)$$

where $\lambda = (p'/p) < 1$, $q = 1 - p$ and $q' = 1 - p'$. Then we set

$$r_c = \{r\} \qquad (4)$$

where $\{x\}$ denotes the smallest integer equal to or greater than x. Similarly, the critical parameter s_c for VT-sampling is found by first solving

$$\delta^s = \frac{1 - P^*}{P^*} , \qquad (5)$$

where $\delta = p'q/pq'$, and then setting

$$s_c = \{s\}. \qquad (6)$$

It is fairly obvious (and can easily be made rigorous) that for any given value of Δ^*, we will have under the least favorable configuration

$$p' = p - \Delta^* \qquad (7)$$

and hence $q' = q + \Delta^*$. Thus only a single parameter (say, p) is required to specify the LF configuration for any given values of Δ^* and P^*.

Some typical curves of r as a function of p are shown in Fig. 1. The salient feature in these curves is that r has a single maximum which occurs at a p-value between $1 - \Delta^*$ and 1, i.e., the maximum (or least favorable) value of $r(p)$ occurs for some value of p, which we denote by p_m, and p_m is close to 1. The error in determining r_c by assuming that $p_m = \bar{p}_0 = 1 - (\Delta^*/2)$ is no greater than 1 and is usually equal to 0. Although we have not proved this point in generality, it is shown in [1] that $p_m \to 1$ as $P^* \to 1$ and we have found no violations for small P^*.

Turning our attention to the parameter s appearing in Eq. (5), we let $p_0 = (p + p')/2$ and write

$$s = \ln(\frac{1 - P^*}{P^*})/\ln(\frac{(2p_0 - \Delta^*)(2q_0 - \Delta^*)}{(2p_0 + \Delta^*)(2p_0 - \Delta^*)}), \qquad (8)$$

where $q_0 = 1 - p_0$ and equation (7) is used for the LF configuration. It is readily verified that s is a unimodal function of p_0 in the interval $(\frac{\Delta^*}{2}, 1 - \frac{\Delta^*}{2})$ with a maximum at $p_0 = 1/2$ or $p = (1 + \Delta^*)/2$. These features of r and s considered as functions of p_0 allow us to specify $p_{LF}(PW)$ and $p_{LF}(VT)$ for the four cases enumerated above. The results for each of these cases are

1. $p_{LF}(PW) = M_u$, $p_{LF}(VT) = M_u$,

2. $p_{LF}(PW) = M_u$, $p_{LF}(VT) = (1 + \Delta^*)/2$

3. $p_{LF}(PW) = M_u$, $p_{LF}(VT) = M_e$

4. $p_{LF}(PW) = \overline{p}_0$, $p_{LF}(VT) = M_e$.

In order to compare results obtained by PW- and VT-sampling we define a loss L by

$$L = (p - p')E(N_B) \qquad (9)$$

where $E(N_B)$ is the expected number of trials on the poorer treatment. That is to say, L is the expected number of failures that could have been prevented through the exclusive use of the better treatment A. One criterion for comparing the effectiveness of the two sampling methods is expressed by the ratio R_L,

$$R_L = L_{PW}/L_{VT} \ . \qquad (10)$$

A second criterion for comparison is expressed by the ratio R_N,

$$R_N = E(N_{PW})/E(N_{VT}) \;, \tag{11}$$

where $E(N)$ is the expected total number of trials needed to reach a decision. The exact expressions for L and $E(N)$ were shown in [1] to be

$$L_{PW} = \frac{(p + 2qr_c)(1 - \lambda^{r_c})(1 - p\lambda - q\lambda^{r_c})}{2(1 - p\lambda - q\lambda^{2r_c})} \tag{12}$$

$$L_{VT} = s_c\left(\frac{1 - \delta^{s_c}}{1 + \delta^{s_c}} \right) \tag{13}$$

$$E(N_{PW}) = \frac{(1 - \lambda^{r_c})(1 - p\lambda - q\lambda^{r_c})}{(1 - \lambda)(1 - p\lambda - q\lambda^{2r_c})} \left[1 + \frac{r}{p}(2 - p - \lambda p)\right] \tag{14}$$

$$E(N_{VT}) = 2L_{VT}/(p - p') \tag{15}$$

where δ is defined after (5). Before presenting our results, let us recall that is was shown in [1] that when $M = (0, 1)$ and we let $P^* \to 1$, then $R_L < 1$ (i.e., PW-sampling is preferred by the L criterion) when

$$p > \frac{3}{4} - \frac{\Delta^*}{8} + o((\Delta^*)^3). \tag{16}$$

It was also shown in the limit $(P^* \to 1)$ that $R_N < 1$ (i.e., PW-sampling is preferred by the N criterion) when

$$\frac{1}{2}(p + p') > \frac{3}{4} - \frac{\Delta^*}{8} + o((\Delta^*)^3). \tag{17}$$

The general results expressed in Equations (16) and (17) were also found to be valid for smaller values of P^*, provided $(\Delta^*)^3$ is small.

In Table 1 we present values of R_L and R_N for case 1. On the basis of the tabulated values one can conclude in case 1 that VT-sampling should be used for small Δ^*, while PW-sampling is otherwise preferable. For $M_u = .25$ the critical value of Δ^* (above which we prefer PW) is approximately .05, while for $M_u = .50$ the critical value of Δ^* is approximately .2. These remarks apply when the comparison criterion is R_L.

Table 2 contains some typical values of R_L and R_N for case 2. The results resemble those obtained in [1] for $M = (0, 1)$, and do not favor either type of sampling uniformly.

Values of R_L and R_N are presented in Table 3 for several sets of parameters for case 3. The data here favor the use of PW-sampling over most of the range.

Table 4 contains a listing of some typical examples for case 4. In this range the use of PW-sampling is advantageous.

To summarize the results of Tables 1-4, we can say that there are no significant differences introduced by restricting M, in the choice of whether to use VT- or PW-sampling. One point of interest is that an increase in Δ^* always favors PW-sampling, and that in case 1, PW-sampling is preferred to VT-sampling for Δ^* sufficiently large. It is conceivable that for M_e in the low part of the range a play-the-loser strategy might lead to better results than either PW- or VT-sampling, but we have not investigated this possibility. We have also not investigated the advantages of the two sampling rules when a different stopping rule is used to terminate testing.

Table 1

Case 1: Values of R_L and R_N for $M_u = .25$ and $.5$

$M_u = .25$

Δ^*	P	$P^* = .75$ R_L	$P^* = .75$ R_N	$P^* = .95$ R_L	$P^* = .95$ R_N	$P^* = .99$ R_L	$P^* = .99$ R_N
.05	.10	1.18	1.22	1.15	1.18	1.12	1.22
	.15	1.15	1.18	1.09	1.13	1.12	1.15
	.20	1.06	1.10	1.04	1.07	1.06	1.09
	.25	.97	.997	.97	.997	.99	1.03
.10	.15	1.46	1.55	1.04	1.10	.97	1.02
	.20	1.47	1.56	.99	1.05	.91	.97
	.25	1.40	1.49	.92	.98	.85	.91
.20	.25	.84	.93	.79	.89	.79	.89

$M_u = .5$

Δ^*	P	$P^* = .75$ R_L	$P^* = .75$ R_N	$P^* = .95$ R_L	$P^* = .95$ R_N	$P^* = .99$ R_L	$P^* = .99$ R_N
.05	.1	1.54	1.58	1.68	1.73	1.72	1.77
	.2	1.55	1.60	1.52	1.56	1.54	1.58
	.3	1.36	1.41	1.36	1.41	1.35	1.40
	.4	1.09	1.13	1.17	1.22	1.17	1.22
	.5	.81	.85	.95	.995	.97	1.01
.10	.2	1.54	1.63	1.32	1.40	1.41	1.50
	.3	1.42	1.52	1.18	1.26	1.24	1.33
	.4	1.17	1.27	1.01	1.10	1.07	1.16
	.5	.91	.99	.82	.90	.89	.98
.20	.3	.73	.82	1.09	1.25	1.08	1.23
	.4	.59	.68	.96	1.11	.94	1.09
	.5	.47	.55	.80	.95	.79	.94
.40	.5	.68	.86	.60	.79	.58	.78

Table 2

Case 2: Values of R_L and R_N for $M_e < (1 + \Delta^*)/2 < M_u < \bar{p}_0$

Δ^*	p	$P^* = .75$		$P^* = .95$		$P^* = .99$	
		R_L	R_N	R_L	R_N	R_L	R_N
		$M_e = .3$, $M_u = .7$					
.05	.4	1.88	1.96	1.69	1.76	1.65	1.72
	.5	1.44	1.51	1.41	1.48	1.38	1.45
	.6	1.01	1.07	1.09	1.16	1.10	1.17
	.7	.64	.87	.77	.84	.82	.88
.10	.4	1.97	2.13	1.53	1.65	1.48	1.60
	.5	1.56	1.70	1.28	1.41	1.24	1.37
	.6	1.14	1.27	1.01	1.13	.998	1.12
	.7	.76	.87	.73	.84	.75	.87
.20	.5	.87	1.02	1.25	1.49	1.14	1.36
	.6	.67	.81	1.01	1.24	.92	1.14
	.7	.49	.61	.75	.98	.70	.92
		$M_e = .1$, $M_u = .9$					
.05	.2	2.89	2.98	2.65	2.73	2.72	2.80
	.4	2.51	2.61	2.13	2.22	2.07	2.16
	.6	1.40	1.48	1.42	1.50	1.39	1.48
	.8	.49	.54	.65	.72	.69	.77
.10	.2	2.56	2.72	2.32	2.46	2.48	2.63
	.4	2.38	2.57	1.85	2.00	1.89	2.04
	.6	1.40	1.56	1.25	1.40	1.28	1.43
	.8	.55	.66	.59	.73	.64	.80
.20	.4	1.55	1.78	1.77	2.05	1.74	2.03
	.6	1.01	1.23	1.25	1.54	1.20	1.49
	.8	.48	.64	.63	.90	.63	.91
.40	.6	1.30	1.77	1.21	1.73	1.04	1.51
	.8	.79	1.18	.71	1.22	.60	1.07

Table 3

Case 3: Values of R_L and R_N for $(1 + \Delta^*)/2 \leq M_e < M_u < \bar{p}_0$

$M_e = .6$, $M_u = .9$

Δ^*	p	$P^* = .75$		$P^* = .95$		$P^* = .99$	
		R_L	R_N	R_L	R_N	R_L	R_N
.05	.7	.89	.96	1.11	1.20	1.09	1.18
	.8	.49	.54	.70	.78	.72	.81
	.9	.21	.25	.35	.43	.37	.45
.10	.7	.93	1.07	1.08	1.25	1.05	1.22
	.8	.55	.66	.69	.85	.70	.87
	.9	.26	.35	.36	.51	.37	.54
.20	.8	.48	.64	.63	.90	.76	1.10
	.9	.28	.42	.36	.61	.42	.76

$M_e = .7$, $M_u = .9$

Δ^*	p	$P^* = .75$ R_L	$P^* = .75$ R_N	$P^* = .95$ R_L	$P^* = .95$ R_N	$P^* = .99$ R_L	$P^* = .99$ R_N
.05	.8	.66	.73	.84	.94	.84	.94
	.9	.27	.32	.41	.50	.43	.53
.10	.8	.55	.66	.83	1.03	.87	1.08
	.9	.26	.35	.42	.60	.45	.65
.20	.9	.84	1.24	.49	.84	.53	.95

Table 4

Case 4: Values of R_L and R_N for $(1+\Delta^*)/2 \le M_e < \bar{p}_0 \le M_u$

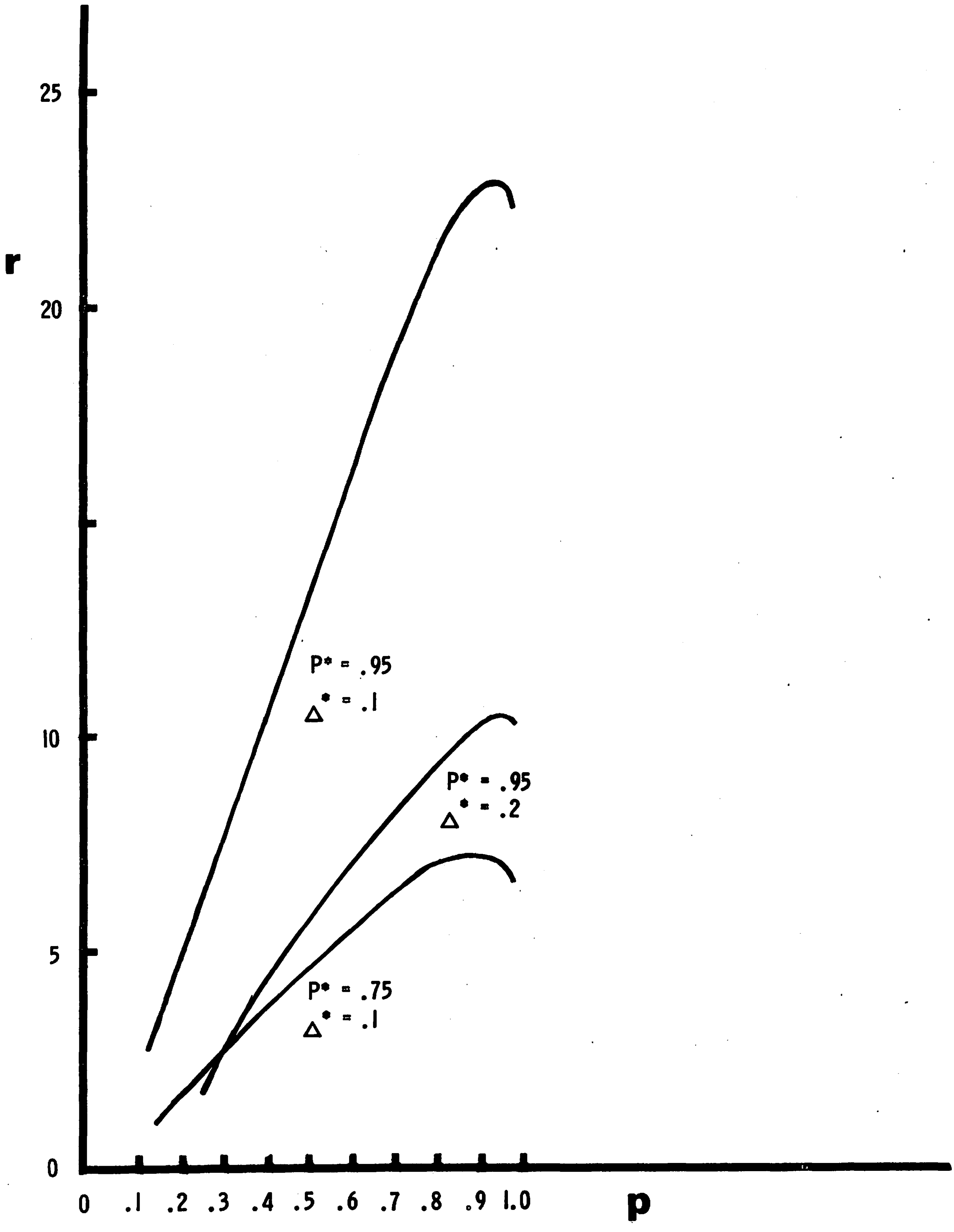
$M_e = .6$, $M_u = 1$

Δ^*	P	$P^* = .75$		$P^* = .95$		$P^* = .99$	
		R_L	R_N	R_L	R_N	R_L	R_N
.05	.7	.80	.87	1.11	1.20	1.12	1.21
	.8	.44	.49	.70	.78	.74	.83
	.9	.19	.23	.35	.43	.38	.47
.10	.7	.76	.87	1.08	1.25	1.08	1.25
	.8	.45	.54	.69	.85	.72	.90
	.9	.22	.29	.36	.51	.38	.55
.20	.8	.48	.64	.63	.90	.81	1.17
	.9	.28	.42	.36	.61	.44	.80

$M_e = .7$, $M_u = 1$

.05	.8	.60	.67	.84	1.09	.86	.97
	.9	.25	.29	.41	.50	.44	.54
.10	.8	.45	.22	.83	1.02	.89	1.11
	.9	.22	.29	.42	.60	.46	.67
.20	.9	.84	1.29	.49	.84	.56	1.01

FIGURE I



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